# TWISTED ALEXANDER POLYNOMIALS WITH THE ADJOINT ACTION FOR SOME CLASSES OF KNOTS

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ABSTRACT. This paper studies the twisted Alexander polynomial with the adjoint action associated to $SL_2(\mathbb{C})$-representations of a knot group for some classes of knots. We explicitly calculate the twisted Alexander polynomial with the adjoint action for torus knots and twist knots. As consequences of these calculations, we obtain the formula of the non-abelian Reidemeister torsion for torus knots in [Du] and a nicer formula of the non-abelian Reidemeister torsion for twist knots than the one in [DHY].

## 1. INTRODUCTION

The Alexander polynomial, the first polynomial knot invariant, was introduced by Alexander in 1928 [Al]. It was later interpreted in terms of Reidemeister torsions by Milnor [Mi] and Turaev [Tu]. The twisted Alexander polynomial, which is a generalization of the Alexander polynomial, was introduced by Lin [Li] for knots in the 3-sphere and by Wada [Wa] for finitely presentable groups. It was also interpreted in terms of Reidemeister torsions by Kitano [Ki] and Kirk and Livingston [KL]. As a consequence of this interpretation, one can calculate certain kinds of Reidemeister torsions of a knot from a finite presentation of its knot group by using Fox's differential calculus.

This paper studies the twisted Alexander polynomial with the adjoint action associated to $SL_2(\mathbb{C})$-representations of a knot group. The adjoint action, denoted by Ad, is the conjugation on the Lie algebra $sl_2(\mathbb{C})$ by elements in the Lie group $SL_2(\mathbb{C})$. Suppose $K$ is a knot and $G_K$ its knot group. For each representation $\rho$ of $G_K$ into $SL_2(\mathbb{C})$, the composition $\mathrm{Ad} \circ \rho$ is a representation of $G_K$ into $SL_3(\mathbb{C})$ and hence, following [Wa], one can define a rational function $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$, called the twisted Alexander polynomial with the adjoint action associated to the $SL_2(\mathbb{C})$-representation $\rho$. The twisted Alexander polynomial $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$ has been calculated for a few knots with small number of crossings [DY]. The purpose of this paper is to calculate $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$ for torus knots and twist knots, see Theorems 3.3 and 4.1. As consequences of these calculations, we obtain the formula of the non-abelian Reidemeister torsion for torus knots in [Du] and a nicer formula of the non-abelian Reidemeister torsion for twist knots than the one in [DHY].

The paper is organized as follows. In Section 2 we review the twisted Alexander polynomial with the adjoint action associated to $SL_2(\mathbb{C})$-representations of a knot group and its relation to the non-abelian Reidemeister torsion. We explicitly calculate the twisted Alexander polynomial with the adjoint action and the non-abelian Reidemeister torsion for torus knots and twist knots in Sections 3 and 4 respectively.

## 2. The twisted Alexander polynomial with the adjoint action

2.1. **Twisted Alexander polynomials.** Let $K$ be a knot and $G_K = \pi_1(S^3 \backslash K)$ its knot group. We choose and fix a presentation

$$G_K = \langle a_1, \ldots, a_\ell \mid r_1, \ldots, r_{\ell-1} \rangle.$$

(This might not be a Wirtinger representation.)

Let $f : G_K \to H_1(S^3 \backslash K, \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$ be the abelianization homomorphism. Here we specify a generator $t$ of $H_1(S^3 \backslash K; \mathbb{Z})$ and denote the sum in $\mathbb{Z}$ multiplicatively. Let us consider a linear representation $\rho : G_K \to SL_k(\mathbb{C})$.

These maps naturally induce two ring homomorphisms $\widetilde{\rho} : \mathbb{Z}[G_K] \to \mathcal{M}(k, \mathbb{C})$ and $\widetilde{f} : \mathbb{Z}[G_K] \to \mathbb{Z}[t^{\pm 1}]$, where $\mathbb{Z}[G_K]$ is the group ring of $G_K$ and $\mathcal{M}(k, \mathbb{C})$ is the matrix algebra of degree $k$ over $\mathbb{C}$. Then $\widetilde{\rho} \otimes \widetilde{f}$ defines a ring homomorphism $\mathbb{Z}[G_K] \to \mathcal{M}(k, \mathbb{C}[t^{\pm 1}])$. Let $F_\ell$ denote the free group on generators $a_1, \ldots, a_\ell$ and $\Phi : \mathbb{Z}[F_\ell] \to \mathcal{M}(k, \mathbb{C}[t^{\pm 1}])$ the composition of the surjection $\mathbb{Z}[F_\ell] \to \mathbb{Z}[G_K]$ induced by the presentation of $G_K$ and the map $\widetilde{\rho} \otimes \widetilde{f} : \mathbb{Z}[G_K] \to \mathcal{M}(k, \mathbb{C}[t^{\pm 1}])$.

Let us consider the $(\ell - 1) \times \ell$ matrix $M$ whose $(i, j)$-component is the $k \times k$ matrix

$$\Phi \left( \frac{\partial r_i}{\partial a_j} \right) \in \mathcal{M} \left( k, \mathbb{Z}[t^{\pm 1}] \right),$$

where $\partial / \partial a$ denotes Fox's differential calculus. For $1 \leq j \leq \ell$, let us denote by $M_j$ the $(\ell - 1) \times (\ell - 1)$ matrix obtained from $M$ by removing the $j$th column. We regard $M_j$ as a $k(\ell - 1) \times k(\ell - 1)$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$. Then Wada's twisted Alexander polynomial of a knot $K$ associated to a representation $\rho : G_K \to SL_k(\mathbb{C})$ is defined to be a rational function

$$\Delta_{K,\rho}(t) = \frac{\det M_j}{\det \Phi(1 - a_j)}$$

and moreover well-defined up to a factor $t^{km}$ $(m \in \mathbb{Z})$, see [Wa].

2.2. **The twisted Alexander polynomial with the adjoint action.** The adjoint action, denoted by Ad, is the conjugation on the Lie algebra $sl_2(\mathbb{C})$ by elements in the Lie group $SL_2(\mathbb{C})$. For $A \in SL_2(\mathbb{C})$ and $g \in sl_2(\mathbb{C})$ we have $\mathrm{Ad}_A(g) = AgA^{-1}$. For each representation $\rho$ of $G_K$ into $SL_2(\mathbb{C})$, the composition $\mathrm{Ad} \circ \rho$ is a representation of $G_K$ into $SL_3(\mathbb{C})$ and hence one can define the twisted Alexander polynomial $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$. We call $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$ the twisted Alexander polynomial with the adjoint action associated to the $SL_2(\mathbb{C})$-representation $\rho$.

**Remark 2.1.** It is known that $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$ coincides with the non-abelian Reidemeister torsion polynomial $\mathcal{T}_{K,\rho}(t)$ [Ki, KL]. As a consequence of this identification, one can calculate the non-abelian Reidemeister torsion $\mathbb{T}_{K,\rho}$ for any longitude-regular $SL_2(\mathbb{C})$-representation $\rho$ from a finite presentation of the knot group of $K$, by applying Fox's differential calculus and the following formula

$$\mathbb{T}_{K,\rho} = -\lim_{t \to 1} \frac{\mathcal{T}_{K,\rho}(t)}{t - 1}$$

in [Ya]. For the detail definition of longitude-regularity, the non-abelian Reidemeister torsion polynomial $\mathcal{T}_{K,\rho}(t)$ and the non-abelian Reidemeister torsion $\mathbb{T}_{K,\rho}$ we refer to [Po, Du, DHY, DY].

In this paper we are interested in the calculation of the twisted Alexander polynomial with the adjoint action associated to non-abelian/irreducible $SL_2(\mathbb{C})$-representations.

## 3. Torus knots

In this section we will calculate the twisted Alexander polynomial with the adjoint action and the non-abelian Reidemeister torsion for torus knots.

Let $K$ be the $(p, q)$-torus knot. The knot group of $K$ is $G_K = \langle c, d \mid c^p = d^q \rangle$. Choose a pair $(r, s)$ of natural numbers satisfying $ps - qr = 1$. Then $\mu = c^{-r}d^s$ is a meridian of $K$. Note that the abelian homomorphism $f : G_K \to H_1(S^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$ sends $c$ and $d$ to $t^q$ and $t^p$ respectively.

A representation $\rho : G_K \to SL_2(\mathbb{C})$ is called irreducible if there is no proper invariant line in $\mathbb{C}^2$ under the action of $\rho(G_K)$. Let $R^{\mathrm{irr}}(G_K)$ be the set of irreducible $SL_2(\mathbb{C})$-representations of $G_K$ and $\hat{R}^{\mathrm{irr}}$ the set of conjugacy classes of representations in $R^{\mathrm{irr}}(G_K)$. According to Johnson, we have the following description of $\hat{R}^{\mathrm{irr}}(G_K)$.

**Proposition 3.1** ([Jo]). $\hat{R}^{\mathrm{irr}}(G_K)$ *consists of* $(p - 1)(q - 1)/2$ *components, which are determined by the following data, denoted by* $\hat{R}^{\mathrm{irr}}_{k,l}(G_K)$:

(1) $0 < k < p$, $0 < l < q$, *and* $k \equiv l \pmod 2$.
(2) *For every* $[\rho] \in \hat{R}^{\mathrm{irr}}_{k,l}(G_K)$, *we have* $\rho(c^p) = \rho(d^q) = (-1)^k I$, $\operatorname{tr}\rho(c) = 2\cos\left(\frac{\pi k}{p}\right)$, $\operatorname{tr}\rho(d) = 2\cos\left(\frac{\pi l}{q}\right)$, *and* $\operatorname{tr}\rho(\mu) \neq 2\cos\pi\left(\frac{rk}{p} \pm \frac{sl}{q}\right)$.

*In particular,* $\hat{R}^{\mathrm{irr}}_{k,l}(G_K)$ *is parametrized by* $\operatorname{tr}\rho(\mu)$ *and has complex dimension one.*

**Remark 3.2.** Proposition 3.1 implies that for every irreducible representation $\rho : G_K \to SL_2(\mathbb{C})$, the matrices $\rho(c)$ and $\rho(d)$ are conjugate to

$$\begin{bmatrix} e^{i\frac{\pi k}{p}} & 0 \\ 0 & e^{-i\frac{\pi k}{p}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e^{i\frac{\pi l}{q}} & 0 \\ 0 & e^{-i\frac{\pi l}{q}} \end{bmatrix}$$

respectively, if the conjugacy class of $\rho$ is contained in $\hat{R}^{\mathrm{irr}}_{k,l}(G_K)$.

**Theorem 3.3.** *Suppose* $[\rho] \in \hat{R}^{\mathrm{irr}}_{k,l}$. *One has*

$$\Delta_{K,\mathrm{Ad}\circ\rho}(t) = \frac{(t^{pq} - 1)^3}{(t^p - 1)(t^q - 1)(t^{2q} - 2(\cos\frac{2\pi k}{p})t^q + 1)(t^{2p} - 2(\cos\frac{2\pi l}{q})t^p + 1)}.$$

*Proof.* By conjugation if necessary, we may assume that $\rho(c) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ and $\rho(d)$ is conjugate to $\begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}$ where $\alpha = e^{i\frac{\pi k}{p}}$ and $\beta = e^{i\frac{\pi l}{q}}$.

Let $\{E, H, F\}$ be the following usual $\mathbb{C}$-basis of the Lie algebra $sl_2(\mathbb{C})$:

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then the adjoint actions of $c$ and $d$ in the basis $\{E, H, F\}$ of $sl_2(\mathbb{C})$ are given the matrices $C = \mathrm{Ad}_{\rho(c)} = \operatorname{diag}(\alpha^2, 1, \alpha^{-2})$ and $D = \mathrm{Ad}_{\rho(d)}$ is conjugate to $\operatorname{diag}(\beta^2, 1, \beta^{-2})$.

We have $\frac{\partial}{\partial c}c^p d^{-q} = 1 + c + \cdots + c^{p-1}$, and hence

$$
\begin{aligned}
\Delta_{K,\mathrm{Ad}\circ\rho}(t) &= \frac{\det\Phi(\frac{\partial}{\partial c}c^p d^{-q})}{\det\Phi(d-1)} \\
&= \frac{\det(I + t^q C + t^{2q}C^2 + \cdots + t^{(p-1)q}C^{p-1})}{\det(t^p D - I)} \\
&= \frac{(1 + \alpha^2 t^q + \cdots + \alpha^{2(p-1)}t^{(p-1)q})(1 + \alpha^{-2}t^q + \cdots + \alpha^{-2(p-1)}t^{(p-1)q})}{(t^{2p} - (\beta^2 + \beta^{-2})t^p + 1)} \\
&\quad \times \frac{1 + t^q + \cdots + t^{(p-1)q}}{t^p - 1}.
\end{aligned}
$$

Since $\alpha^{2p} = 1$, we have

$$
1 + \alpha^2 t^q + \cdots + \alpha^{2(p-1)}t^{(p-1)q} = \frac{(\alpha^2 t^q)^p - 1}{\alpha^2 t^q - 1} = \frac{t^{pq} - 1}{\alpha^2 t^q - 1}.
$$

Similarly,

$$
1 + \alpha^{-2} t^q + \cdots + \alpha^{-2(p-1)}t^{(p-1)q} = \frac{t^{pq} - 1}{\alpha^{-2} t^q - 1}.
$$

Hence

$$
\Delta_{K,\mathrm{Ad}\circ\rho}(t) = \frac{(t^{pq} - 1)^3}{(t^p - 1)(t^q - 1)(t^{2q} - (\alpha^2 + \alpha^{-2})t^q + 1)(t^{2p} - (\beta^2 + \beta^{-2})t^p + 1)}
$$

The theorem follows since $\alpha^2 + \alpha^{-2} = 2\cos(\frac{2\pi k}{p})$ and $\beta^2 + \beta^{-2} = 2\cos(\frac{2\pi l}{q})$. $\qquad\square$

**Remark 3.4.** The proof of Theorem 3.3 is similar to that of [KM, Theorem 1.1].

It is known that for a torus knot, any irreducible $SL_2(\mathbb{C})$-representation $\rho$ is longitude-regular so that one can define the non-abelian Reidemeister torsion $\mathbb{T}_{K,\rho}$, see [Po, Du].

**Corollary 3.5.** *Suppose* $[\rho] \in \hat{R}_{k,l}^{\mathrm{irr}}$. *One has*

$$
\mathbb{T}_{K,\rho} = -\frac{p^2 q^2}{16\sin^2\left(\frac{\pi k}{p}\right)\sin^2\left(\frac{\pi l}{q}\right)}.
$$

*Proof.* By [Ya, Theorem 3.1.2], we have $\mathbb{T}_{K,\rho} = -\lim\limits_{t\to 1}\frac{\Delta_{K,\mathrm{Ad}\circ\rho}(t)}{t-1}$. The corollary then follows from Proposition 3.3. $\qquad\square$

**Remark 3.6.** Corollary 3.5 was obtained in [Du, Section 6.2] by a different method. Note that our torsion is the inverse of the one in [Du].

## 4. TWIST KNOTS

In this section we will calculate the twisted Alexander polynomial with the adjoint action and the non-abelian Reidemeister torsion for twist knots.

Let $K = J(2, 2n)$ be the twist knot in the notation of [HS]. The knot group of $K$ is $G_K = \langle w^n a = b w^n \rangle$ where $w = ba^{-1}b^{-1}a$ and $a, b$ are meridians. A representation $\rho : G_K \to SL_2(\mathbb{C})$ is called non-abelian if $\rho(G_K)$ is a non-abelian subgroup of $SL_2(\mathbb{C})$. Taking conjugation if necessary, we can assume that $\rho$ has the form

$$
(4.1) \qquad \rho(a) = \begin{bmatrix} \sqrt{s} & 1/\sqrt{s} \\ 0 & 1/\sqrt{s} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} \sqrt{s} & 0 \\ -\sqrt{s}\,u & 1/\sqrt{s} \end{bmatrix}
$$

where $(s, u) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the Riley equation $\phi_K(s, u) = 0$, see [Ri].

Let $\gamma = \operatorname{tr} \rho(w) = 2 + 2u - \frac{u}{s} - su + u^2$. Note that

$$\phi_K(s, u) = (s + s^{-1} - 1 - u)\frac{\xi_+^n - \xi_-^n}{\xi_+ - \xi_-} - \frac{\xi_+^{n-1} - \xi_-^{n-1}}{\xi_+ - \xi_-}$$

where $\xi_\pm$ are eigenvalues of $\rho(w)$, i.e. $\xi_+\xi_- = 1$ and $\xi_+ + \xi_- = \gamma$, see [DHY].

**Theorem 4.1.** *Suppose $\rho$ is a non-abelian representation of the form* (4.1)*. One has*

$$\Delta_{K,\operatorname{Ad}\circ\rho}(t) = \frac{t-1}{(y+2-x^2)(y^2 - yx^2 + x^2)}\left(nt^2 + \frac{(2n-1)y^2 + yx^2 - 2nx^2(x^2 - 2)}{y^2 - yx^2 + 2x^2}t + n\right)$$

*where $x = \operatorname{tr}\rho(a) = \operatorname{tr}\rho(b) = \sqrt{s} + 1/\sqrt{s}$ and $y = \operatorname{tr}\rho(ab^{-1}) = u + 2$.*

*Proof.* Recall that $\{E, H, F\}$ be the following usual $\mathbb{C}$-basis of the Lie algebra $sl_2(\mathbb{C})$:

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is easy to see that the adjoint actions of $a$ and $b$ in the basis $\{E, H, F\}$ of $sl_2(\mathbb{C})$ are given the following matrices:

$$A = \operatorname{Ad}_{\rho(a)} = \begin{bmatrix} s & -2 & -s^{-1} \\ 0 & 1 & s^{-1} \\ 0 & 0 & s^{-1} \end{bmatrix}, \quad B = \operatorname{Ad}_{\rho(b)} = \begin{bmatrix} s & 0 & 0 \\ su & 1 & 0 \\ -su^2 & -2u & s^{-1} \end{bmatrix}.$$

We focus on the case $n > 0$. The case $n < 0$ is similar. We have $\frac{\partial}{\partial a}w^n a w^{-n} b^{-1} = w^n(1 + (1-a)(1 + w^{-1} + \cdots + w^{-(n-1)})(a^{-1} - a^{-1}b))$, and hence

$$\begin{aligned} \Delta_{K,\operatorname{Ad}\circ\rho}(t) &= \frac{\det \Phi(\frac{\partial}{\partial a}w^n a w^{-n} b^{-1})}{\det \Phi(b-1)} \\ &= \frac{\det(I + (I - tA)(I + W^{-1} + \cdots + W^{-(n-1)})(t^{-1}A^{-1} - A^{-1}B))}{\det(tB - I)} \end{aligned}$$

where $W = \operatorname{Ad}_{\rho(w)}$.

Let $\Omega = I + W^{-1} + \cdots + W^{-(n-1)}$. Then

$$\Delta_{K,\operatorname{Ad}\circ\rho}(t) = \frac{\det(I + (I - tA)\Omega(t^{-1}A^{-1} - A^{-1}B))}{\det(tB - I)}.$$

To calculate $\Omega$, we need to diagonalize the $SL_3(\mathbb{C})$-matrix $W$.

The $SL_2(\mathbb{C})$-matrix $\rho(w)$ can be diagonalized by

$$Q = \begin{bmatrix} u + 1 - s^{-1} & u + 1 - s^{-1} \\ 1 - su - \xi_+ & 1 - su - \xi_- \end{bmatrix}.$$

Explicitly, $Q^{-1}\rho(w)Q$ is the diagonal matrix $\operatorname{diag}(\xi_+, \xi_-)$.

Set $\alpha = 1 - su - \xi_+$, $\beta = 1 - su - \xi_-$ and $\delta = u + 1 - s^{-1}$. With respect to the basis $\{E, H, F\}$ of $sl_2(\mathbb{C})$, the matrix of the adjoint action of $Q$ becomes as follows:

$$P = \operatorname{Ad}_Q = \frac{1}{\alpha - \beta}\begin{bmatrix} -\delta & 2\delta & \delta \\ \alpha & -(\alpha + \beta) & -\beta \\ \alpha^2/\delta & -2\alpha\beta/\delta & -\beta^2/\delta \end{bmatrix}.$$

Since $P^{-1}WP$ is the diagonal matrix $\mathrm{diag}(\xi_+^2, 1, \xi_-^2)$, by direct calculations we have

$$
\begin{aligned}
\Omega &= I + W^{-1} + \cdots + W^{-(n-1)} \\
&= P \, \mathrm{diag}(\xi_-^{n-1}X, n, \xi_+^{n-1}X) \, P^{-1}
\end{aligned}
$$

$$
(4.2)= \frac{1}{(\alpha - \beta)^2}
\begin{bmatrix}
-2\alpha\beta n + d_5 X & 2\delta(-(\alpha+\beta)n + d_3 X) & \delta^2(2n - d_1 X) \\
\frac{\alpha\beta}{\delta}((\alpha+\beta)n - d_3 X) & (\alpha+\beta)^2 n - 2\alpha\beta d_1 X & \delta(-(\alpha+\beta)n + d_2 X) \\
(\frac{\alpha\beta}{\delta})^2(2n - d_1 X) & \frac{2\alpha\beta}{\delta}((\alpha+\beta)n - d_2 X) & -2\alpha\beta n + d_4 X
\end{bmatrix}
$$

where

$$
d_1 = \xi_-^{n-1} + \xi_+^{n-1}, \;\; d_2 = \alpha\,\xi_-^{n-1} + \beta\,\xi_+^{n-1}, \;\; d_3 = \alpha\,\xi_+^{n-1} + \beta\,\xi_-^{n-1},
$$

$$
d_4 = \alpha^2 \xi_-^{n-1} + \beta^2 \xi_+^{n-1}, \;\; d_5 = \alpha^2 \xi_+^{n-1} + \beta^2 \xi_-^{n-1},
$$

$$
X = \frac{\xi_+^n - \xi_-^n}{\xi_+ - \xi_-}, \;\; Y = \frac{\xi_+^{n-1} - \xi_-^{n-1}}{\xi_+ - \xi_-}.
$$

**Proposition 4.2.** *One has*

$$
\Omega = \frac{1}{s^2 u(1 - 2s + s^2 - su)(-4s + u - 2su + s^2 u - su^2)}
\begin{bmatrix}
\omega_{11} & \omega_{12} & \omega_{13} \\
\omega_{21} & \omega_{22} & \omega_{23} \\
\omega_{31} & \omega_{32} & \omega_{33}
\end{bmatrix}
$$

*where*

$$
\begin{aligned}
\omega_{11} &= s^2 u \big\{ (1 - 2s + s^2 - su)(2 - 4s + 2s^2 + u - 6su + s^2 u - 4s^3 u \\
&\quad - su^2 + 3s^2 u^2 - s^3 u^2 + s^4 u^2 - s^3 u^3)X^2 - 2ns(-1 + s + su)^2 \big\}, \\
\omega_{12} &= -2su(-1 + s + su)\big\{ (1 - 2s + s^2 - su)(-1 - 3s^2 + 2su - s^2 u + s^3 u - s^2 u^2)X^2 \\
&\quad - ns(-1 + 2s + s^2 + su) \big\}, \\
\omega_{13} &= -(-1 + s + su)^2 \big\{ (1 - 2s + s^2 - su)(-2s + u - su + s^2 u - su^2)X^2 - 2ns^2 \big\}, \\
\omega_{21} &= s^2 u^2(-1 + s + su)\big\{ (1 - 2s + s^2 - su)(-1 - 3s^2 + 2su - s^2 u + s^3 u - s^2 u^2)X^2 \\
&\quad - ns(-1 + 2s + s^2 + su) \big\}, \\
\omega_{22} &= -su\big\{ 2(1 - 2s + s^2 - su)(-1 + s + su)^2(-2s + u - su + s^2 u - su^2)X^2 \\
&\quad - nsu(-1 + 2s + s^2 + su)^2 \big\}, \\
\omega_{23} &= -u(-1 + s + su)\big\{ (1 - 2s + s^2 - su)(-1 + s + su) \\
&\quad (-3s + s^2 + u - 2su + s^2 u - su^2)X^2 - ns^2(-1 + 2s + s^2 + su) \big\}, \\
\omega_{31} &= -s^2 u^2(-1 + s + su)^2 \big\{ (1 - 2s + s^2 - su)(-2s + u - su + s^2 u - su^2)X^2 - 2ns^2 \big\}, \\
\omega_{32} &= 2su^2(-1 + s + su)\big\{ (1 - 2s + s^2 - su)(-1 + s + su) \\
&\quad (-3s + s^2 + u - 2su + s^2 u - su^2)X^2 - ns^2(-1 + 2s + s^2 + su) \big\}, \\
\omega_{33} &= u(-1 + s + su)\big\{ (1 - 2s + s^2 - su)(-2s^2 + 2s^3 + 4su - 9s^2 u + 3s^3 u \\
&\quad - u^2 + 4su^2 - 9s^2 u^2 + 3s^3 u^2 + 2su^3 - 4s^2 u^3 + s^3 u^3 - s^2 u^4)X^2 \\
&\quad - 2ns^3(-1 + s + su) \big\}.
\end{aligned}
$$

*Proof.* We need the following lemma which can be checked easily.

**Lemma 4.3.** *One has*

$$
\begin{aligned}
d_1 &= 2X - \gamma Y, \\
d_2 &= (1 - su)(2X - \gamma Y) - \gamma X + (\gamma^2 - 2)Y, \\
d_3 &= (1 - su)(2X - \gamma Y) - \gamma X + 2Y, \\
d_4 &= (1 - su)^2(2X - \gamma Y) - 2(1 - su)(\gamma X - (\gamma^2 - 2)Y) + (\gamma^2 - 2)X - \gamma(\gamma^2 - 3)Y, \\
d_5 &= (1 - su)^2(2X - \gamma Y) - 2(1 - su)(\gamma X - 2Y) + (\gamma^2 - 2)X - \gamma Y.
\end{aligned}
$$

We now prove Proposition 4.2. Since $(s, u) \in \mathbb{C}^* \times \mathbb{C}$ satisfies the Riley equation

$$
\phi_K(s, u) = (s + s^{-1} - 1 - u)\frac{\xi_+^n - \xi_-^n}{\xi_+ - \xi_-} - \frac{\xi_+^{n-1} - \xi_-^{n-1}}{\xi_+ - \xi_-} = 0,
$$

we have $(s + s^{-1} - 1 - u)X - Y = 0$. The proposition then follows by direct calculations and by applying Identity (4.2), Lemma 4.3, and Identity $Y = (s + s^{-1} - 1 - u)X$. $\quad\square$

To proceed, we need the following lemma.

**Lemma 4.4.** *One has*

$$
X^2 = \frac{1}{1 - (s + s^{-1} - 1 - u)\gamma + (s + s^{-1} - 1 - u)^2}.
$$

*Proof.* It is easy to check that

$$
X^2 - \gamma XY + Y^2 = \left(\frac{\xi_+^n - \xi_-^n}{\xi_+ - \xi_-}\right)^2 - (\xi_+ + \xi_-)\left(\frac{\xi_+^n - \xi_-^n}{\xi_+ - \xi_-}\right)\left(\frac{\xi_+^{n-1} - \xi_-^{n-1}}{\xi_+ - \xi_-}\right) + \left(\frac{\xi_+^{n-1} - \xi_-^{n-1}}{\xi_+ - \xi_-}\right)^2 = 1.
$$

Since $Y = (s + s^{-1} - 1 - u)X$, the lemma follows. $\quad\square$

We now complete the proof of Theorem 4.1. By Proposition 4.2, Lemma 4.4, and direct calculations, we have

$$
\begin{aligned}
\Delta_{K,\mathrm{Ad}\circ\rho}(t) &= \frac{\det(I + (I - tA)\Omega(t^{-1}A^{-1} - A^{-1}B))}{\det(tB - I)} \\
&= \frac{s(t-1)}{(-1 + s - u)(1 - 2s + s^2 - su)(-1 + s + su)(-4s + u - 2su + s^2u - su^2)t^3} \\
&\quad \times \big\{ ns(-4s + u - 2su + s^2u - su^2)t^2 + (2n - 2s + 4ns - 4ns^2 \\
&\quad - 2s^3 + 4ns^3 + 2ns^4 - su + 2s^2u - 8ns^2u - s^3u + s^2u^2 - 2ns^2u^2)t \\
&\quad + ns(-4s + u - 2su + s^2u - su^2) \big\} \\
&= \frac{t-1}{(-2 + 1/s + s - u)(-2 + 1/s + s - 2u + u/s + su - u^2)t^3} \\
&\quad \times \Big\{ nt^2 + \frac{1}{-4 - 2u + u/s + su - u^2}\big( -4n + 2n/s^2 - 2/s + 4n/s - 2s + 4ns \\
&\quad + 2ns^2 + 2u - 8nu - u/s - su + u^2 - 2nu^2 \big)t + n \Big\}.
\end{aligned}
$$

Theorem 4.1 follows by substituting $s + 1/s = x^2 - 2$, $s^2 + 1/s^2 = x^4 - 4x^2 + 2$ and $u = y - 2$ into the above identity. $\quad\square$

**Corollary 4.5.** *Suppose $\rho$ is a longitude-regular $SL_2(\mathbb{C})$-representation. One has*

$$\mathbb{T}_{K,\rho} = \frac{-1}{(y+2-x^2)(y^2-yx^2+x^2)} \left( \frac{(2n-1)y^2 + yx^2 - 2nx^2(x^2-2)}{y^2 - yx^2 + 2x^2} + 2n \right),$$

*where $x = \operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$ and $y = \operatorname{tr}\rho(ab^{-1})$.*

*Proof.* By [Ya, Theorem 3.1.2], $\mathbb{T}_{K,\rho} = -\lim_{t \to 1} \dfrac{\Delta_{K,\operatorname{Ad}\circ\rho}(t)}{t-1}$. Corollary 4.5 then follows from Theorem 4.1. $\qquad\square$

**Remark 4.6.** The non-abelian Reidemeister torsion for twist knots was calculated in [DHY]. However, our formula in Corollary 4.5 is nicer.

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